Fall 2003 Math 308/501–502
1 Introduction
1.1 Background
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Summary

When modeling applications, equating the rate at which a quantity changes (a derivative) with an application-specific way of formulating the rate of change leads to a **differential equation (DE)**; i.e., an equation that contains some derivative(s) of an unknown function.

Recall the concept of **proportionality**. With k a constant (often positive), we have the following mathematical relationships (among many).

MATH	ENGLISH
$y = kx$	y is proportional to x
$y = kx^2$	y is proportional to the square of x
$y = kxz$	y is proportional to the product of x and z
$y = k/x^3$	y is inversely proportional to the cube of x
$dP/dt = kP$	the rate of change of P is proportional to P

Here are some terms used in the study of differential equations.

- **independent variable**: a variable in a DE upon which the unknown function depends; often t (time) or x (when the context is geometrical).
- **dependent variable**: a variable whose derivative appears in a DE; usually y, occasionally x or some other letter.
- **coefficients**: multipliers of the unknown function or its derivatives, either constant or depending on independent variable(s) only.
- **ordinary differential equations (ODEs)**: differential equations containing only ordinary derivatives.
- **partial differential equations (PDEs)**: differential equations containing only partial derivatives.
- **scalar differential equation**: a single DE.
- **system of differential equations**: a collection of two or more DEs.
- **linear ODE**: an ordinary differential equation in which the unknown function and its derivatives appear in additive combinations of their first powers only. More precisely, a DE that may be put into the form

$$\sum_{k=0}^{n} a_k(x) \frac{d^k y}{dx^k} = F(x)$$

- **nonlinear ODE**: an ordinary differential equation that is not linear.

In this course, we shall study ODEs exclusively and linear ones primarily.

Text and lab manual nomenclature

References to textbook exercises have the form ***NSS4-p/x*** or ***NSS4: p/x*** or more simply ***p/x***. Here ***p*** is the page number and ***x*** is the exercise number. (The ***NSS4***, of course, refers to the 4th edition of Nagle/Saff/Snider.)

Textbook exercise directories are of the form ***n*PPP*x##***, where PPP is the page number and ## is the exercise number. (We use these for organizational purposes.)

References to lab manual exercises have the form ***M-p/x*** or ***P-p/x***. Here ***p*** is the page number and ***x*** is the exercise in your lab manual (Polking).

Lab manual exercise directories are of the form ***p*PPP*x##***, where PPP is the page number and ## is the exercise number. (We use these for organizational purposes.)

Hand Examples

TERMINOLOGY PROBLEMS

NSS4-5/2 or 5/2

Classify the differential equation $\dfrac{d^2y}{dx^2} - 2x\dfrac{dy}{dx} + 2y = 0$.

Solution

This is a second-order variable-coefficient linear ODE. Its independent variable is x; its dependent variable is y.

5/4

Classify the differential equation $\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0$.

Solution

This is a second-order constant-coefficient linear PDE. Its independent variables are x and y. Its dependent variable is u.

6/6

Classify the differential equation $\dfrac{dx}{dt} = (4 - x)(1 - x)$.

Solution

This one is a bit tricky. Let's expand the differential equation.

$$\frac{dx}{dt} = 4 - 5x + x^2$$

We see that this is a first-order nonlinear ODE, due to the fact that the dependent variable x occurs to the second power. The DE's independent variable is t; its dependent variable is x.

Example A

Classify the differential equation $y''' + 3y'' + 3y' + y = 0$.

Solution

This is a third-order constant-coefficient linear ODE. Its dependent variable is y. Its independent variable, which is not explicitly specified, may be taken as t or x (anything other than y).

MODELING PROBLEMS

In the following exercises, model the application with a differential equation. Rates are with respect to time t.

Example B

The rate of growth of a population is inversely proportional to the square root of the population.

Solution

Let P be the population and let t represent time. Then we have $dP/dt = k/\sqrt{P}$.

Example C

The rate of decay of a given radioactive substance is proportional to the amount of the substance remaining.

Solution

Let y be the amount of the substance at time t. Then $dy/dt = -ky$. Here k is a positive proportionality constant and the rate of growth is negative, signifying radioactive decay; i.e., as time goes on, there is less of the substance. (An alternative formulation would be $dy/dt = ay$, where a is a negative proportionality constant.)

Example D

A potato that has been cooking for some time is removed from a heated oven. The room temperature of the kitchen is 65°F. The rate at which the potato cools is proportional to the difference between the temperature of the potato (the object whose temperature is changing) and room temperature (the temperature of the object's surroundings).

Solution

Let T be the temperature of the potato at time t. Then $dT/dt = -k(T - 65)$ with k positive or $dT/dt = a(T - 65)$ with a negative.

Example E

A particle moves along the x-axis, its position from the origin at time t given by $x(t)$. A single force acts on the particle that is proportional to but opposite the object's displacement. Use Newton's law to derive a differential equation of the object's motion.

Solution

Let $v = dx/dt$ be the particle's velocity and $a = d^2x/dt^2$ its acceleration. Newton's second law tell us that force equals mass times acceleration, or $F = ma$. Since $F = -kx$ with k positive, we have

$$m\frac{d^2x}{dt^2} = -kx \quad \text{or} \quad \frac{d^2x}{dt^2} = -\frac{k}{m}\,x$$